Predictability of velocity and temperature fields in intermittent turbulence

1993 J. Phys. A: Math. Gen. 26 6943

(http://iopscience.iop.org/0305-4470/26/23/034)

J. Phys. A: Math. Gen. 26 (1993) 6943–6960. Printed in the UK

Predictability of velocity and temperature fields in intermittent turbulence

A Crisanti†, M H Jensen‡, G Paladin§ and A Vulpiani†

† Dipartimento di Fisica, Università 'La Sapienza', P le A Moro 2, I-00185 Roma, Italy
‡ NORDITA, Blegdamsvej 17, Copenhagen Ø, DK-2100 Denmark
§ Dipartimento di Fisica, Università dell'Aquila, I-67100, L'Aquila, Italy

Received 5 April 1993, in final form 28 July 1993

Abstract. We discuss the relation between predictability and the sensitive dependence on initial conditions in turbulent flows. The dependence on the Reynolds number of the maximum Lyapunov exponent of the flow is evaluated within the multifractal model. The temporal intermittency of the degree of chaos is found to localize on the degrees of freedom corresponding to high wavenumbers. We discuss the consequences for the mechanism for the growth of small perturbations. We show that intermittency causes long tails in the distribution of the predictability times both in the velocity field and in a passively advected scalar field, though there is no simple relation between these two times. Numerical evidence of this picture is provided within the framework of a cascade model for velocity and temperature fields in fully developed turbulence.

1. Introduction

Prediction is an old and classical problem in fluid mechanics. According to Laplace (1814), it is, in principle, possible to predict the state of a system at any time $t > 0$, if one knows its evolution laws and the initial condition. After the contribution of Poincaré, and more recently of Lorenz (1963), it is now well understood that predictability has severe limitations in the presence of deterministic chaos because of the exponential divergence of the distance between two initially close trajectories.

Typically, in nonlinear systems an uncertainty $\delta\boldsymbol{x}(0)$ on the state of the system at time $t = 0$ increases as

$$|\delta\boldsymbol{x}(t)| \simeq |\delta\boldsymbol{x}(0)|\,\mathrm{e}^{\lambda t}. \tag{1.1}$$

The growing rate λ is called the maximum Lyapunov exponent (Benettin *et al* 1980a, b). Consequently, if $|\delta\boldsymbol{x}(0)| = \delta_0$ and one accepts $\delta_{\max}$ as the maximum tolerance on the knowledge of the state of the system, (1.1) implies that the system is predictable up to a time

$$T \sim \frac{1}{\lambda}\ln\left(\frac{\delta_{\max}}{\delta_0}\right). \tag{1.2}$$

Equation (1.2) shows that the *predictability time* is proportional to λ^{-1}. In fact, the dependence on the precision of the initial and final time is very weak, only logarithmic and can be neglected.

Equation (1.2) gives only an approximate first answer to the problem of prediction since it does not take into account some important features of chaotic systems:

(i) In general there are fluctuations in the degree of chaos, so that beyond (1.1) it is necessary to introduce the effective Lyapunov exponent $\gamma_t(\tau)$ (Eckmann and Procaccia 1986, Paladin *et al* 1986):

$$|\delta x(t+\tau)| \simeq |\delta x(\tau)| \exp[\gamma_t(\tau)\, t] \qquad (1.3)$$

where $\gamma_t(\tau)$ depends on t and τ.

(ii) Equation (1.1) neglects the possible influence of the many degrees of freedom. In some problems it is rather natural to have an uncertainty at the initial time only on some degrees of freedom while the main interest, as time proceeds, is focused on other degrees of freedom. For example, in weather forecasting there are errors on the small scale of the initial state while one wishes to know the large-scale behaviour.

(iii) Equations (1.1) and (1.3) are only valid for small δ_0.

In this paper we study the statistics of the predictability time T and its relation with the intermittency of the degree of chaos in turbulence and chaotic dynamical systems.

In section 2 we briefly review the basic concepts for the characterization of the effective Lyapunov exponent in terms of the generalized Lyapunov exponents (Benzi *et al* 1985).

In section 3 we study a model, whose evolution laws are given by a set of ordinary differential equations, the so-called shell model (Yamada and Ohkitani 1987, 1988), which mimics many of the physically relevant features of turbulence. In section 4, introducing a perturbation on the small scale, we look at the 'butterfly effect', i.e. the cascade and growth of a perturbation from small to large scales.

In section 5 we compute, in the framework of the multifractal model of turbulence (Benzi *et al* 1984, Parisi and Frisch 1985), the scaling of λ and μ, related to the average and variance of γ respectively, as a function of the Reynolds number Re. The scaling exponents are obtained from the eddy turnover time and the Kolmogorov length. These results are compared with numerical simulation of the shell model.

In section 6, we discuss the probability distribution of T in the shell model for different Re and the relation with the intermittency. By considering this relation in a more general context we show that the main qualitative features are not peculiar to the particular chosen system. For a weakly intermittent system, i.e. small fluctuations of γ, the probability distribution of T is roughly Gaussian while for strong intermittency it exhibits long tails.

The 'standard' theory of predictability in fully developed turbulence (Leith and Kraichnan 1972, Lilly 1973) gives, neglecting intermittency,

$$T \sim T_0 \qquad (1.4)$$

where T_0 is the typical eddy turnover time at large scale. This is in strong disagreement with our result showing that T decreases for increasing Re. We discuss this point in section 7. Finally, in section 8 studying suitable shell models, we present the result for the predictability of a passive scalar field θ, e.g. the temperature. In this case, there are two predictability times: T^u for the velocity field and T^θ for the passive scalar. In general, T^θ is smaller than T^u and, sometimes, one may have a long T^u, corresponding to laminar situation for the velocity field, with T^θ close to its typical value. This phenomenon is similar to 'Lagrangian chaos' where chaotic motion of a test particle is observed even in the absence of Eulerian turbulence (Aref 1984, Crisanti *et al* 1991).

2. Dynamical intermittency: general features

One of the most relevant features of chaotic systems is their unpredictability in the temporal evolution. The distance between two trajectories starting at close initial conditions increases exponentially in time. In general, the exponential rate of growth depends on the initial time. One can observe regular motion for long times, interrupted by randomly distributed bursts of strong chaotic behaviour. This phenomenon is called *temporal intermittency*. The degree of chaos is usually measured by the typical exponential rate of growth, given by the largest Lyapunov exponent. It measures the growth of a small disturbance after a very long time. However, finite-time fluctuations may be very important.

A quantitative description of intermittency can be obtained from the *generalized* Lyapunov exponents $L(q)$ which take into account the finite-time properties of the flow (Fujisaka 1983, Benzi *et al* 1985). Let us consider a generic dynamical system described by the set n of differential equations

$$\dot{x}(t) = F[x, t]. \tag{2.1}$$

The response of the system to a perturbation $\delta x(\tau)$ of its state at time τ after a time t, is measured by the error growth rate

$$R_\tau(t) \equiv \frac{|z(\tau + t)|}{|z(\tau)|} \simeq \frac{|\delta x(\tau + t)|}{|\delta x(\tau)|} \tag{2.2}$$

where z is the tangent vector whose evolution is given by

$$\dot{z}(t) = \mathbf{A}(t) z(t) \tag{2.3}$$

obtained by linearizing the evolution equation (2.1) along the trajectory $x(t)$. The matrix $\mathbf{A}$ is the Jacobian matrix of (2.1), $A_{ij} = \partial F_i / \partial x_j$.

By definition, the maximal Lyapunov exponent is

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \langle \ln R_\tau(t) \rangle \tag{2.4}$$

where the angular brackets denote the time average along the trajectory. The Oseledec theorem (Oseledec 1969) ensures that the average in (2.4) can be removed, since for almost all initial conditions $\lambda = \lim_{t \to \infty} \ln R_\tau(t)/t$.

The Lyapunov exponent λ does not characterize the fluctuations in the response of the system to a perturbation. A direct calculation of the probability distribution of R is, in general, not feasible. A convenient way is to reconstruct it from its moments. One thus introduces the generalized Lyapunov exponents

$$L(q) = \lim_{t \to \infty} \frac{1}{t} \ln \langle R_\tau(t)^q \rangle. \tag{2.5}$$

It is easy to verify that

$$\lambda = \left. \frac{\mathrm{d}L(q)}{\mathrm{d}q} \right|_{q=0}. \tag{2.6}$$

In general, $L(q)$ is a convex function of q. Linear behaviour for $L(q) = \lambda q$ denotes absence of intermittency.

Knowledge of $L(q)$ gives information on the large fluctuations in $R_\tau(t)$ for finite t, as one can see by defining the effective Lyapunov exponent $\gamma_t(\tau)$ as

$$R_\tau(t) \simeq \exp[\gamma_t(\tau)\, t] \qquad \text{for } t \gg 1. \tag{2.7}$$

The trajectories of length t can be classified according to the value of γ. The generalized Lyapunov exponents are obtained by averaging over all possible values of γ

$$\langle R_\tau(t)^q \rangle = \int \mathrm{d}\gamma\; P_t(\gamma)\, \mathrm{e}^{\gamma q t} \tag{2.8}$$

where $P_t(\gamma)$ is the probability density of having a trajectory of length t with an effective Lyapunov exponent in the interval $[\gamma, \gamma + \mathrm{d}\gamma]$. For large t, a sensible *ansatz* for $P_t(\gamma)$ is (Eckmann and Procaccia 1986, Paladin *et al* 1986)

$$P_t(\gamma)\, \mathrm{d}\gamma = \rho(\gamma)\, \mathrm{e}^{-S(\gamma)\, t}\, \mathrm{d}\gamma \qquad S(\gamma) \geqslant 0 \tag{2.9}$$

where $\rho(\gamma)$ is a smooth function. From the Oseledec theorem it follows that $S(\gamma) = 0$ only for $\gamma = \lambda$. By inserting (2.9) into (2.8), and performing the integral by saddle-point integration for large t, one has

$$\langle R_\tau(t)^q \rangle = \int \mathrm{d}\gamma\; \rho(\gamma)\, \mathrm{e}^{[\gamma q - S(\gamma)]\, t} \sim \mathrm{e}^{t\, L(q)} \tag{2.10}$$

with

$$L(q) = \max_\gamma [q\gamma - S(\gamma)]. \tag{2.11}$$

The Legendre transform (2.11) shows that each value of q selects a particular class of trajectories with γ given by

$$q = \left.\frac{\mathrm{d}S(\gamma)}{\mathrm{d}q}\right|_\gamma . \tag{2.12}$$

By inverting the Legendre transform, the convex envelope of $S(\gamma)$ is obtained from $L(q)$.

In general situations, the distribution of R for small deviations from the typical value $\exp(\lambda t)$ is close to a log-normal distribution (Paladin and Vulpiani 1987a)

$$P[R(t)] \propto \frac{1}{R(t)\sqrt{2\pi \mu t}} \exp\left[-\frac{(\ln R(t) - \lambda t)^2}{2\mu t}\right] \tag{2.13}$$

where

$$\mu = \lim_{t\to\infty} \frac{1}{t}[\langle (\ln R(t))^2 \rangle - \langle \ln R(t) \rangle^2] = \lim_{t\to\infty} t[\langle \gamma_t^2 \rangle - \langle \gamma_t \rangle^2].$$

The moments of the distribution (2.13) give

$$L(q) = \lambda q + \tfrac{1}{2}\mu q^2. \tag{2.14}$$

This form of $L(q)$ is valid only for small q. The above results can also be stated by saying that if $L(q)$ admits the expansion (2.14) for small q, then the probability distribution for small deviations of γ from the typical value is close to a log-normal.

We stress that when the moments $\langle R_\tau(t)^q \rangle$ grow more than exponentially with q for $q \gg 1$, the probability distribution is not uniquely determined by $L(q)$. Nevertheless, (2.11) still gives information on the small deviations of γ from the typical value λ. It is worth stressing that the approximation (2.14) for small q yields the parabolic shape $S(\gamma) = (\gamma - \lambda)^2/2\mu$ for γ close to λ. In some cases, when central limit arguments cannot be applied, this approximation can fail (Grassberger *et al* 1988, Crisanti *et al* 1992).

From our considerations, it follows that at a first level, intermittency can be characterized by two parameters: λ and μ. The former gives the typical value, the latter measures the variance of the γ-fluctuations. The ratio $\mu/\lambda = 1$ delimits the borderline between weak and strong intermittency. The maximum of $P[R(t)]$, equation (2.13), is reached for

$$\overline{R}(t) = \mathrm{e}^{\lambda t(1-\mu/\lambda)} \tag{2.15}$$

so that for $\mu/\lambda > 1$ intermittency gives drastic corrections to the 'mean field' result obtained by estimating the response by the value which maximizes the probability distribution. For a discussion on the relevance of intermittency in geophysics see, e.g., Benzi and Carnevale (1989).

In fully developed turbulence the term 'intermittency' is usually used to indicate strong fluctuations in the energy dissipation (Monin and Yaglom 1975). It is responsible for the corrections to the scaling laws of the classical phenomenological theory of Kolmogorov (1941). In turbulent flows there is an energy transfer from large towards small scales where dissipation, due to molecular friction, overwhelms the non-linear cascade mechanism. The transfer is hierarchical, in the sense that a disturbance on a certain scale receives its energy from larger scale disturbances, and transfers it to smaller scale disturbances. At the end of the cascade the energy is transformed into heat by molecular friction.

Assuming a constant rate of nonlinear energy transfer from larger to smaller scales, one obtains the classical Kolmogorov results. In fact, dimensional analysis suggests that the Navier–Stokes equations have singular velocity gradients in the limit of infinite Reynolds numbers, i.e. the velocity difference on a scale $\ell = |\boldsymbol{\ell}|$ goes as $\delta v(\ell) \equiv |\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{\ell}) - \boldsymbol{v}(\boldsymbol{x})| \sim \ell^h$ with $h = 1/3$. As a consequence of the uniformity of the energy transfer rate, it follows that in the inertial range the velocity structure functions scale as

$$\langle \delta v(\ell)^q \rangle \propto \ell^{\zeta_q} \qquad \text{with } \zeta_q = q/3. \tag{2.16}$$

There is much experimental (Anselmet *et al* 1984) and numerical evidence (Vincent and Meneguzzi 1991) for the presence of strong spacetime fluctuations in the energy transfer and dissipation. They lead to a whole spectrum of possible singularities h and, in particular, to nonlinear behaviour of ζ_q as a function of q. Over the last few years, several phenomenological approaches have been proposed to explain these corrections (Mandelbrot 1975); the fractal description of turbulence has assumed a central role (Parisi and Frisch 1985, Benzi *et al* 1984, Paladin and Vulpiani 1987a).

One could conjecture that the intermittent behaviour must be intimately related to the dynamical properties of the time evolution of the velocity field ruled by the Navier–Stokes equations. To this goal, from the very beginning of the modern theories of turbulence, simple dynamical models have been introduced to capture the main features of the full Navier–Stokes equations in terms of a 'small' number of equations (Desnyansky and Novikov 1974).

3. A shell model

We consider the shell model proposed by Yamada and Ohkitani (1987, 1988), which is defined as follows. The Fourier space is divided into N shells. Each shell, denoted by k_n with $n = 1, 2, \ldots, N$, consists of the wavenumbers $K_0 2^n < k \leqslant K_0 2^{n+1}$, where K_0 is a constant. The velocity difference over a length scale $\approx k_n^{-1}$ is given by a complex variable u_n. The energy is $E = \sum_n |u_n|^2/2$ and its power spectrum $E(k_n) = \langle |u_n|^2 \rangle/(2k_n)$. The Navier–Stokes equations are thus replaced by $2N$ ordinary differential equations

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu k_n^2\right) u_n = \mathrm{i}\,(a_n u_{n+1}^* u_{n+2}^* + b_n u_{n-1}^* u_{n+1}^* + c_n u_{n-1}^* u_{n-2}^*) + f\,\delta_{n,4} \tag{3.1}$$

where ν is the viscosity, and f is a forcing, here on the fourth mode.

There are two main qualitative differences from the Navier–Stokes equations:

(i) k is a scalar;

(ii) there are only nearest and next-nearest neighbour interactions among the shells.

In particular, (i) implies the loss of all the effects due to the geometrical structures. Point (ii) is rather sensible, as long as the energy cascade is local in Fourier space, with exponentially decreasing interactions among shells.

The coefficients of the nonlinear terms follow by demanding energy and phase-space conservation in the unforced inviscid limit, i.e. $\nu = f = 0$:

$$\begin{aligned} &a_n = k_n \qquad b_n = -\tfrac{1}{2}(k_{n-1}) \qquad c_n = -\tfrac{1}{2}(k_{n-2}) \\ &b_1 = b_N = c_1 = c_2 = a_{N-1} = a_N = 0. \end{aligned} \tag{3.2}$$

For $\nu = f = 0$, (3.1) has an unstable fixed point given by the Kolmogorov law $u_n \propto k_n^{-1/3}$. The time evolution generated by (3.1) exhibits chaotic behaviour on a strange attractor in the $2N$-dimensional phase space, with a maximum Lyapunov exponent roughly proportional to $\nu^{-1/2}$ (see section 5). The Reynolds number can be easily changed over several orders of magnitude just by varying the kinematic viscosity ν and the number of shells N.

Numerical integration of (3.1) reveals an energy spectrum $E(k)$ which scales in the inertial range as $k^{-\alpha}$, with an exponent $\alpha = 1 + \zeta_2 \simeq 1.7$ slightly different from the value 5/3 of the Kolmogorov theory. The exponents ζ_q are not linear in q, and can be fitted by the random β-model formula (Benzi *et al* 1984)

$$\zeta_q = q/3 - \ln_2[1 - x + x(\tfrac{1}{2})^{1-q/3}] \qquad x = 0.12 \tag{3.3}$$

where only two possible kinds of fragmentation are assumed in the cascade process: either vorticity sheets, with probability x, or space filling disturbances, as in the Kolmogorov theory, with probability $1 - x$. The value $x = 0.12$ is very close to $x = 0.125$ used to fit the experimental data of Anselmet *et al* (1984). The intermittency of the energy dissipation exhibited by the model is consistent with the multifractal approach.

The corrections to the Kolmogorov predictions can be connected to the temporal intermittency in the dynamical evolution, since the energy bursts are observed to interrupt quiescent laminar periods, with a corresponding increase in the effective Lyapunov exponent. A solution of (3.1) spends most of the time in the laminar phase where energy dissipation $\epsilon = \nu \sum_n k_n^2 |u_n(t)|^2$ is lower than the average value $\overline{\epsilon}$. In the same time, the total energy $E(t) = \frac{1}{2}\sum_n |u_n(t)|^2$ of the system slowly increases up to the arrival of a sudden burst. At this point, there is a regime of very high dissipation ($\epsilon \gg \overline{\epsilon}$) with a fast energy decrease.

In the plane E, ϵ, one observes 'cycles' of slow energy charge and fast discharge around the Kolmogorov fixed point $\overline{E}$, $\overline{\epsilon}$. During a cycle there are small variations in the energy while ϵ changes over several orders of magnitude.

For our purposes, it is more important to analyse the behaviour in the tangent space which describes the growth of small perturbations. In the laminar phase, the eigenvector associated with the maximum Lyapunov exponent spreads around the forced wavenumber and over the whole inertial range, while in the chaotic regime it concentrates on the dissipative shells (Jensen *et al* 1991). One thus has a localization of the largest instability on the dissipative scales with consequent excitation of the disturbance at large wavenumbers during simultaneous chaotic and energy bursts.

In the rest of the paper, we shall analyse in detail the consequences of the above scenario for the growth of a disturbance and estimating the predictability time.

4. The butterfly effect: inverse cascade of perturbations

As briefly discussed in the introduction, the sensitive dependence on initial conditions makes long-term forecasting impossible. For example, Ruelle (1979) remarked that thermal fluctuations in the atmosphere produce observable changes on a scale of centimetres after only one minute. One thus expects that after one or two weeks, the earth's atmospheric circulation would be unpredictable, even if the exact evolution equations were known. This is the so-called *butterfly effect*, after the words of Lorenz: *a butterfly moving its wings over Brazil might cause the formation of a tornado over Texas*. Here we estimate the predictability times using the shell model introduced in the previous section.

The state of the shell model at a time t will be denoted by $\boldsymbol{u}(t) \equiv \{u_1(t), \ldots, u_n(t)\}$. We work in the regime of fully developed turbulence, $Re > 10^5$. At a certain time t a perturbed state $\boldsymbol{u}'(t)$ is produced by adding a small increment ϵ to the velocity component in some of the shells, and the distance between the two trajectories is defined as

$$D(\tau) = |\boldsymbol{u}(t+\tau) - \boldsymbol{u}'(t+\tau)|. \tag{4.1}$$

If the evolution is chaotic $D(\tau)$ grows exponentially in time, i.e.

$$\langle \ln D(\tau) \rangle \approx \lambda\, \tau \qquad \text{for } \tau \gg 1 \tag{4.2}$$

where the average is over several perturbations.

To study the butterfly effect we perturb the model at high wavenumbers, close to the dissipative cut-off given by the Kolmogorov length. To gain more insight into the growth of the disturbance, we introduce the difference on the nth shell at a time τ of the two fields

$$|\delta u_n(\tau)|^2 = |u_n(t+\tau) - u_n'(t+\tau)|^2. \tag{4.3}$$

The exponential growth of $|\delta u_n(\tau)|^2$ is triggered by a large energy burst localized at the small length scales and associated with a sharp increase in the instantaneous Lyapunov exponent. After such a burst, the value of $|\delta u_n(\tau)|^2$ increases with τ at smaller and smaller k_n, so that the initial disturbance localized on small scales propagates towards lower k_n by a sort of inverse cascade, as shown in figure 1(a). The disturbance eventually reaches the

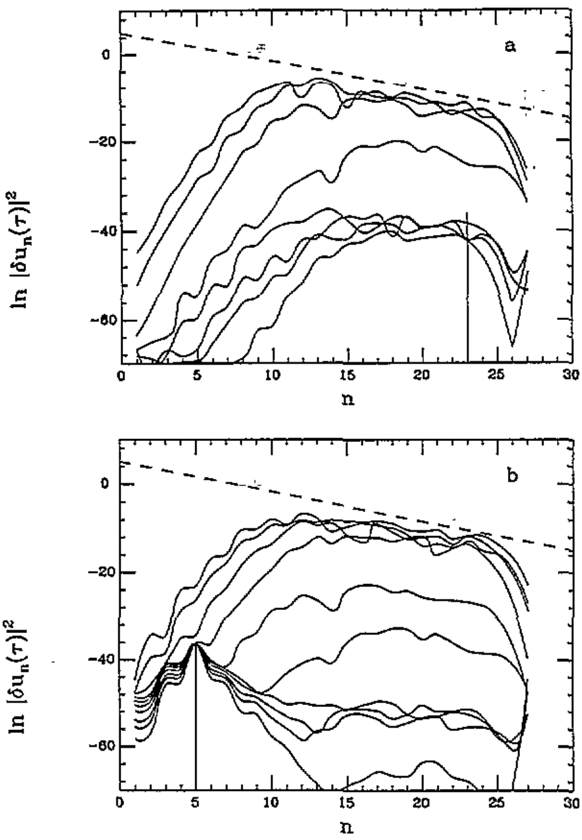

Figure 1. Plot of $\ln|\delta u_n(\tau)|^2$ as a function of n for different values of τ, for a model with 27 shells, $f = 5(1+i) \times 10^{-3}$, $K_0 = 0.05$, and $Re = 2 \times 10^9$. In (a) the perturbation at $\tau = 0$ is performed on the shell $n = 23$, with $|\delta u_{23}(0)| = 10^{-8}$ and $|\delta u_n(0)| = 0$ for $n \neq 23$ as shown by the vertical line. The following three curves are calculated with time interval 0.2 in τ and the remaining curves with time interval 0.4. Note a rather big jump between the fourth and the fifth curve which is due to the occurrence of an energy dissipation burst at that time. In (b) the perturbation at $\tau = 0$ is performed on large scale (the shell $n = 5$) as indicated by the vertical line. The time interval between the curves are the same as in (a). The disturbance becomes small scale before the exponential growth starts. The broken lines show the scaling law $|\delta u_n|^2 \simeq k_n^{-2/3}$ of the Kolmogorov theory.

beginning of the inertial range affecting the flow at large scales: the 'butterfly' disturbance has grown to macroscopic scales.

If the disturbance is not initially localized on the Kolmogorov scale, we do not find an exponential growth of $|\delta u_n(\tau)|^2$ for any shell k_n, until the disturbance has spread all the way down to the small (dissipative) scales. The time needed by this 'precursor' disturbance to move from small to large wavenumbers is very fast, of the order of λ^{-1}. After this time interval, the exponential growth is again triggered by a chaotic burst localized on large k_n, see figure 1(b), leading to the inverse cascade previously discussed for figure 1(a). In other words the butterfly effect always stems from a small scale close to the dissipative cut-off even when the perturbation is performed on a large scale.

5. Energy and dynamical intermittency

In this section we discuss the relation between the multifractality of energy dissipation and dynamical intermittency on the growth of a disturbance in the velocity field $\boldsymbol{v}(\boldsymbol{x})$ (Crisanti *et al* 1993).

In three-dimensional fully developed turbulence, the maximum Lyapunov exponent should be roughly proportional to the inverse of the smallest characteristic time of the system, the turnover time τ of eddies of the size of the Kolmogorov length η. We can introduce, in terms of the spatial average of the energy dissipation $\overline{\epsilon}$ and of the typical large length scale of the system L, the corresponding typical velocity $V = (\overline{\epsilon}\, L)^{1/3}$ and time $T_0 = L/V = (L^2/\overline{\epsilon})^{1/3}$. The turnover time of an eddy of size ℓ is, by dimensional counting,

$$\tau(\ell) \simeq \frac{v(\ell)}{\ell} \simeq T_0 \ell^{1-h} \tag{5.1}$$

where h is the Hölder exponent of the velocity difference in the eddy,

$$v(\ell) \equiv |\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{v}(\boldsymbol{x})| \sim V\,\ell^h \tag{5.2}$$

and $\ell = r/L$ is the non-dimensional scaling parameter. The nonlinear transfer of energy is stopped at the Kolmogorov scale η where viscosity ν is able to compete with the convective term, $\nu \sim \eta\, v(\eta)$. It follows that the viscous cut-off vanishes as a power of the Reynolds number $Re = V\,L/\nu$ (Paladin and Vulpiani 1987b),

$$\eta(h) \sim L\, Re^{-1/(1+h)}\,. \tag{5.3}$$

These dimensional relations imply that the maximum Lyapunov exponent should scale with Re as

$$\lambda \sim \frac{1}{\tau(\eta)} \sim \frac{1}{T_0} Re^{\alpha} \qquad \text{with} \qquad \alpha = \frac{1-h}{1+h} \tag{5.4}$$

since it should be proportional to the shortest characteristic time of the system, i.e. the turnover time of the smallest eddy of size η. In the Kolmogorov theory $h = 1/3$ for all space points, so that $\alpha = 1/2$, as first pointed out by Ruelle (1979).

The presence of quiescent quasi-laminar periods should change the chaotic features of the fluid flow. The intermittency of energy dissipation can be described by introducing a spectrum of singularities h. In the multifractal approach, the probability that the velocity difference scales with an exponent h is assumed to be

$$P_\ell(h) \sim \ell^{3-D(h)} \tag{5.5}$$

where the function $D(h)$ is given by the Legendre transform of the velocity structure exponent defined in (2.16):

$$\zeta_q = \min_h [hq - D(h) + 3]. \tag{5.6}$$

Multifractality also implies the existence of a spectrum of viscous cut-offs, since each h selects a different damping scale according to (5.3), and hence a spectrum of turnover times. To obtain the maximum Lyapunov exponent, we have to integrate $\tau(h)^{-1}$, given by (5.1) at the scale $\ell = \eta(h)/L$, over the h-distribution $P_\ell(h)$:

$$\lambda \sim \int \tau(h)^{-1} P_\ell(h) \, \mathrm{d}h \sim \frac{1}{T_0} \int \left(\frac{\eta}{L}\right)^{h-D(h)+2} \mathrm{d}h. \tag{5.7}$$

From (5.3) the viscous cut-off vanishes in the limit $Re \to \infty$ and the integral can be estimated by the saddle-point method,

$$\lambda \sim \frac{1}{T_0} Re^{\alpha} \qquad \text{with} \qquad \alpha = \max_h \left[\frac{D(h) - 2 - h}{1 + h}\right]. \tag{5.8}$$

The value of α depends on $D(h)$. By using the function $D(h)$ obtained by fitting the exponents ζ_q with the random beta model (3.3) we find $\alpha = 0.459\ldots$, slightly smaller than the Ruelle prediction $\alpha = 0.5$.

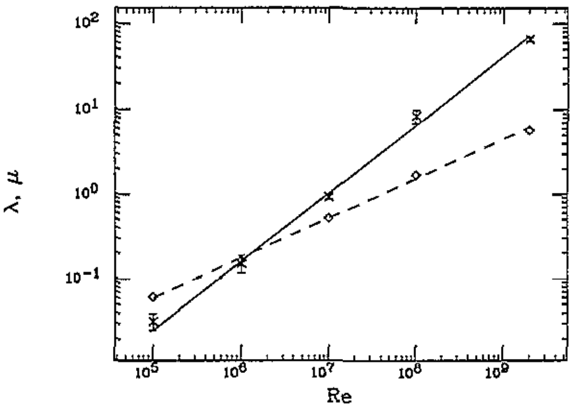

Figure 2. The Lyapunov exponent, λ (diamonds) and μ (crosses), as a function of the Reynolds numbers from a shell model calculation with $N = 27$ shells, $f = 5(1+\mathrm{i}) \times 10^{-3}$ and $K_0 = 0.05$. The broken line is the multifractal prediction $\lambda \sim Re^{\alpha}$ with $\alpha = 0.459$, where the function $D(h)$ is given by the random beta model fit of the ζ_p exponents. The full line indicates $\mu \sim Re^{w}$ with $w = 0.8$.

In figure 2 it is shown the Lyapunov exponent as a function of the Reynolds number for the shell model with $N = 27$ shells. The different values of Re are obtained by changing the value of the viscosity ν. The correction to the Ruelle prediction $\lambda \sim Re^{1/2}$ is well evident and agrees with (5.8). The variance μ of the finite-time fluctuations is also reported. We remember that λ and μ give the main characterization of the distribution of the effective Lyapunov exponent, and that the value $\mu/\lambda = 1$ separates weak from strong intermittency.

The numerical data show that the variance diverges as

$$\mu(Re) \sim Re^{w} \tag{5.9}$$

with $w \simeq 0.8$. The variation in the fluctuations of the effective Lyapunov exponent can be computed from the multifractal spectrum of $\tau(h)^{-2}$. Noting that

$$\gamma_\tau(t) = \frac{1}{\tau} \int_t^{t+\tau} \gamma_0(t') \, \mathrm{d}t' \tag{5.10}$$

an explicit calculation leads to

$$\mu \sim \int_0^\infty \langle (\gamma_0(t+t') - \lambda)\,(\gamma_0(t) - \lambda)\rangle\, \mathrm{d}t' \sim \langle (\gamma_0 - \lambda)^2\rangle \int_0^\infty C(t')\,\mathrm{d}t' \quad (5.11)$$

where $C(t')$ is the normalized correlation function of the effective Lyapunov exponent

$$C(t') = \langle (\gamma_0(t) - \lambda)\,(\gamma_0(t+t') - \lambda)\rangle / \langle (\gamma_0(t) - \lambda)^2\rangle \quad (5.12)$$

which has the same qualitative behaviour of the energy dissipation correlation function (Jensen *et al* 1991). We define the characteristic time

$$t_c = \int_0^\infty C(t')\,\mathrm{d}t' \sim T_0\, Re^{-z} \quad (5.13)$$

which is assumed to vanish as a power of Re. The quantity $\langle (\gamma_0(t) - \lambda)^2\rangle$ can be estimated by repeating the arguments used for λ, so that

$$\langle \gamma_0^2\rangle \sim \int \tau(h)^{-2}\, P_\eta(h)\,\mathrm{d}h \sim \frac{1}{T_0^2}\, Re^y \qquad \text{with} \qquad y = \max_h \left[\frac{D(h) - 1 - 2h}{1+h}\right] = 1.$$

The result $y = 1$ is model independent, since $\langle \gamma_0^2\rangle \sim Re\,\overline{\epsilon}$, where the spatial average of the energy dissipation density $\overline{\epsilon}$ is a finite quantity independent of Re. The fact that $\langle \gamma_0^2\rangle \gg \lambda^2$ at high Re implies

$$\mu \sim \langle \gamma_0^2\rangle\, t_c \simeq \frac{1}{T_0}\, Re^w$$

with $w = 1 - z$. In the shell model this relation is satisfied with $z \simeq 0.2$.

We stress that in the absence of intermittency one may expect that $t_c \sim \lambda^{-1}$, and thus $z \simeq 1/2$. As a consequence, $z \simeq 0.2$ indicates that the presence of quiescent periods in the turbulent activity is much more relevant for the decay rate of time correlations than for the Lyapunov exponent.

Although it is sensible to expect $w > 1/2$ in real turbulent fluids, we cannot ignore the fact that $w \simeq 0.8$ could be due to the particular form of the time correlations in the shell model.

The basic qualitative feature of these results is just the dynamical counterpart of the multifractality of energy dissipation in three-dimensional space. In generic chaotic systems a lower bound to t_c is given by λ^{-1}. It follows that $w \geqslant 1/2$ and $w > \alpha$, implying that μ/λ diverges as $Re \to \infty$, and, hence, dynamical intermittency occurs.

6. Statistics of T and its relation with intermittency

The predictability time for the shell model is defined as follows. Consider two initial realizations $\boldsymbol{u}$ and $\boldsymbol{u}'$, at time $t = 0$

$$u'_n(0) = u_n(0) + \delta u_n(0)$$

with $\delta u_n(0) \neq 0$ only for $n = n^*$, $n^* + 1$ where n^* corresponds to the Kolmogorov length. The predictability time T is defined as the maximum time t such that

$$|\delta u_4(t)|^2 + |\delta u_5(t)|^2 < \delta_{\max}. \quad (6.1)$$

By changing the initial condition, e.g. by taking the same trajectory at different times, the above computation can be repeated many times and the probability distribution of T obtained. The time T is not constant but strongly dependent on the degree of chaos: if the system undergoes an energy burst, T is very short. On the other hand, if the system is in a laminar period, T can be very large. Figure 3 shows the probability distribution function (PDF) of T for two different values of Re. At $Re \simeq 10^6$ we observe a rather peaked PDF with an almost Gaussian shape. For larger values of Re ($Re \simeq 2 \times 10^9$) the distribution acquires an exponential tail, indicating the possibility of large excursions in the value of T, depending on whether the system is in a turbulent or in a purely laminar period.

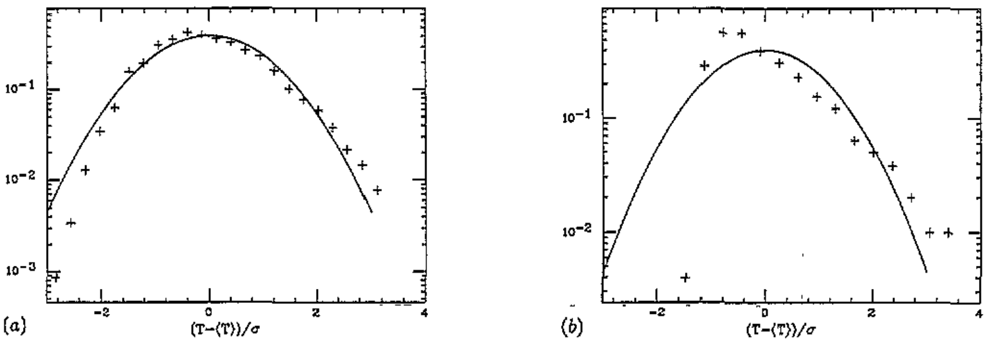

Figure 3. Rescaled probability distribution functions PDF of the predictability time T for the shell model: $\sigma\, P(T)$ against $(T - \langle T \rangle)/\sigma$ for (a): $Re = 10^6$ and (b) $Re = 2 \times 10^9$. The respective average values are $\langle T \rangle = 84.0$, 6.32 and the standard deviations $\sigma = [\langle (T - \langle T \rangle)^2 \rangle]^{1/2}$ are 22.2 and 3.16, respectively. The full curve is the standard Gaussian.

Furthermore, the typical predictability time T_t, i.e. the value of T where the PDF reaches it maximum, is very dependent on the Lyapunov exponent and hence on the Reynolds number. In the shell model, the typical predictability time decreases as a power of Re, and at increasing Re the occurrence of large values of $(T - T_t)/T_t$ is more and more likely.

We want to stress that many features of the above scenario do not depend on the values of the threshold $\delta_{\max}$, the value of the perturbation at the initial time and the precise definition of T, e.g. instead of using (6.1), we can define T as the maximum time such that

$$|\delta u_4|^2 < \delta_{\max}.$$

The gross features of the probability distributions shown in figure 3 do not depend on the particular dynamical system considered but only on the degree of intermittency measured by μ/λ: when $\mu/\lambda \gg 1$ the probability distribution of the predictability time has a long exponential tail, while for $\mu/\lambda \leqslant 1$ it is very peaked. A long exponential tail also appears in the Lorentz model with r slightly larger than $r_c \simeq 166.07$, or in the Pomeau–Manneville map (Pomeau and Manneville 1980), near the intermittent transition, as μ/λ increases, see figure 4.

It seems reasonable to conclude that the mechanism for the occurrence of exponential tails is not an artifact of the shell model, but rather a robust feature of highly intermittent systems. A simple argument shows the relation between the PDF of T and the fluctuations of the effective Lyapunov exponent γ. As a first rough approximation we can assume that (Fujisaka 1983, Benzi *et al* 1985)

$$\ln R(t) = \lambda\, t + \sqrt{\mu}\, w(t) \tag{6.2}$$

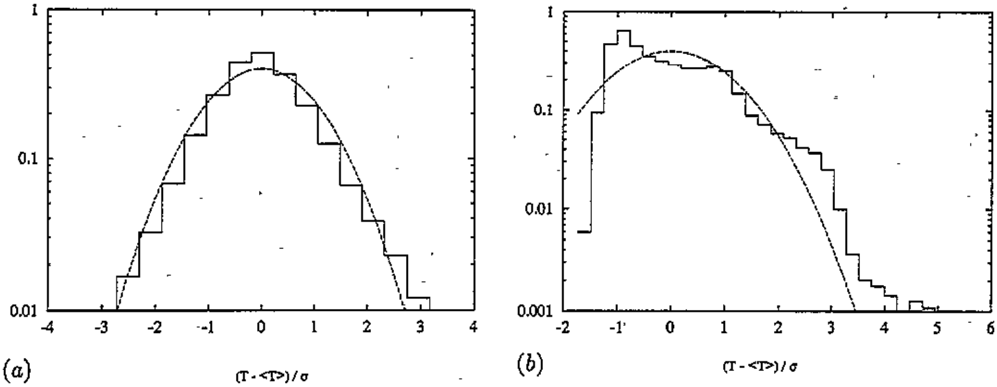

Figure 4. Rescaled probability distribution functions PDF of the predictability time T for the Lorenz model: $\sigma\, P(T)$ against $(T - \langle T\rangle)/\sigma$. In (*a*) $r = 28$ and $\mu/\lambda \simeq 5 \times 10^{-2}$. In (*b*) $r = 166.3$ and $\mu/\lambda \simeq 3.3$. The average values are: $\langle T\rangle = 10.84, 8.24$ and $\sigma = 1.77, 4.44$, respectively. The broken curve is the standard Gaussian.

where $w(t)$ is a Wiener process with $w(0) = 0$ and

$$\langle w(t)\rangle = 0 \qquad \langle w(t)\, w(t')\rangle = \min[t, t'].$$

By this approximation the predictability problem is reduced to a first exit problem: T is the largest time such that

$$\sqrt{\mu}\, w(t) < \ln(\delta_{\max}/\delta_0) - \lambda t. \tag{6.3}$$

This is a standard problem in stochastic process theory, whose solution (Burgers 1974) gives the PDF of T:

$$P(T) = \frac{|\ln(\delta_{\max}/\delta_0)|}{\sqrt{2\pi\mu T^3}} \exp\left[-\frac{(\lambda T - \ln(\delta_{\max}/\delta_0))^2}{2\mu T}\right]. \tag{6.4}$$

For $\lambda = 0$ the PDF is normalized. For $\lambda > 0$ one has a non-zero probability that (6.3) holds at any time, this probability is $1 - \exp(-2\lambda \ln(\delta_{\max}/\delta_0)/\mu)$.

For small value of μ/λ the PDF is almost Gaussian and the mean value of T is close to the most probable value T_t given by the maximum of (6.4)

$$T_t \simeq \frac{1}{\lambda} \ln(\delta_{\max}/\delta_0). \tag{6.5}$$

In contrast for $\mu/\lambda \gg 1$ the PDF exhibits an asymmetric 'triangular shape' and

$$T_t \simeq \frac{1}{3\mu}(\ln(\delta_{\max}/\delta_0))^2. \tag{6.6}$$

In the approximation (6.2) one neglects correlations between $\gamma_t(0) - \lambda$ and $\gamma_{t'}(0) - \lambda$ for $t \neq t'$. This is reasonable only for times much larger than λ^{-1}. Since $\ln(\delta_{\max}/\delta_0)$ is not large, T/λ cannot be too large and the previous argument is not very accurate. The qualitative behaviour predicted by the approximation (6.2) is, however, confirmed both in the shell model and the Lorenz system.

7. Classical predictability approach in turbulence

The first attempt to estimate T in turbulence was due to Lorenz (1969), see also Lilly (1973), based on physical arguments, and to Leith and Kraichnan (1972), based on closure approximations. The basic ingredients of the Lorenz approach are the following. The time $\tau(k)$ for a perturbation at wavenumber $2k$ to induce a complete uncertainty on the velocity field on the wavenumber k, is assumed to be proportional to the typical eddy turnover time at scale k:

$$\tau(k) \sim 1/kv_k \tag{7.1}$$

where v_k is the typical velocity difference at scale $1/k$

$$v_k^2 \sim \int_k^{2k} E(k)\,\mathrm{d}k. \tag{7.2}$$

In the Kolmogorov theory, $E(k) \sim k^{-5/3}$ and $\tau(k) \sim k^{-2/3}$. The predictability time for an uncertainty to propagate from the Kolmogorov scale $\eta \sim k_K^{-1}$ to the scale of the energy containing eddies $L_0 \sim k_0^{-1}$, is given by

$$T \simeq \sum_{n=0}^{N} \tau(2^n k_K) \tag{7.3}$$

where $k_K \sim Re^{3/4}\,k_0$ and $N = \ln_2(k_K/k_0) \sim \ln Re$. From (7.1), (7.2) and (7.3) one has

$$T \sim T_0 \sim L_0/v_0. \tag{7.4}$$

Closure approximations, where one still uses dimensional arguments, give the same results.

Equation (7.4) gives T independent of the Reynolds number. This result does not change by taking into account the correction to the scaling due to intermittency of the energy dissipation.

In the above arguments many characteristic times are involved so that (7.4) strongly depends on the physical mechanism for the inverse cascade of the perturbation.

Our results are different, only one characteristic time, the eddy turnover time at the Kolmogorov scale, is involved. This leads, neglecting the intermittency effects, to $T \sim Re^{-1/2}$, and hence a strong dependence on the Reynolds number.

One may be tempted to conclude that this is an artifact of the shell model since, on physical grounds, it could seem rather strange that predictability on a large scale is related to the properties of the Kolmogorov scale. However, recent direct simulations (Kida and Ohkitani 1992) for three-dimensional turbulence with 340^3 modes, gives a scenario similar to that observed in the shell model. In fact, although the energy is concentrated on a large scale and entropy on a small scale, Kida and Ohkitani observed that the exponential rate for the growth of energy and entropy disturbance are the same and given by the Lyapunov exponent.

We close this section with a short remark. In our analysis, as well as in the classical approach of Lorenz, Lilly, Leith and Kraichnan, one considers the effects of a perturbation on a small scale on the field at a large scale. On the other hand, even for very high Reynolds number, there can exist well defined coherent structures, e.g. vortex tubes, which move while roughly maintaining their shape. In this case, if the interest is only in some qualitative behaviour, one should reformulate the problem of predictability. For example, a reasonable question is the prediction of the centre and the orientation of the vortex tubes. In this case one could hope to have a long predictability time. Of course, this problem cannot be studied within the shell model, where all the spatial structures are neglected.

8. Predictability for passive scalars

We conclude by discussing the problem of predictability for passive scalars θ governed by the evolution equation

$$\partial_t \theta + \boldsymbol{v} \cdot \nabla \theta = D \Delta \theta + S \tag{8.1}$$

where S is an external forcing. Now we have to consider the evolution of two different realizations of the velocities and the scalar fields $\boldsymbol{Y} = (\boldsymbol{v}, \theta)$ and $\boldsymbol{Y}' = (\boldsymbol{v}', \theta')$ such that at $t = 0$ the $\boldsymbol{Y} \simeq \boldsymbol{Y}'$. As above, the difference at $t = 0$ is taken to be small scale. According to section 6 we can define two times, T^u and T^θ, for the predictability of velocity and scalar at large scale.

Following the ideas of section 3 a shell model can be introduced to describe the behaviour of a temperature field passively advected by a velocity field (Jensen *et al* 1992). The equations for the velocity are those described in section 3, equations (3.1) and (3.2). In a similar way, i.e. by projecting (8.1) on the shells in the Fourier space, we obtain for the passive scalar

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}t} + Dk_n^2\right)\theta_n = \mathrm{i}[&e_n\,(u_{n-1}^*\theta_{n+1}^* - u_{n+1}^*\theta_{n-1}^*) + g_n\,(u_{n-2}^*\theta_{n-1}^* + u_{n-1}^*\theta_{n-2}^*) \\ &+ h_n\,(u_{n+1}^*\theta_{n+2}^* + u_{n+2}^*\theta_{n+1}^*)] + S\delta_{n,4}\end{aligned} \tag{8.2}$$

where θ_n are complex variables and S is the external forcing, here on the fourth mode. The conservation of phase space for $\nu = D = f = S = 0$ is automatically satisfied by the absence of diagonal terms proportional to u_n, θ_n on the right-hand side of (3.1) and (8.2). The coefficients of the nonlinear terms follow from demanding the conservation of $\sum_n |\theta_n|^2$ in the absence of forcing and molecular diffusion, i.e. $S = D = 0$. This leads to a possible choice

$$\begin{aligned}&e_n = \tfrac{1}{2}k_n \qquad g_n = -\tfrac{1}{2}k_{n-1} \qquad h_n = -\tfrac{1}{2}k_{n+1} \\ &e_1 = e_N = g_1 = g_2 = h_{N-1} = h_N = 0.\end{aligned} \tag{8.3}$$

In the unforced, inviscid limit, i.e. $\nu = D = f = S = 0$, equations (3.1) and (8.2) have an unstable fixed point given by the Obukhov–Corrsin scaling $u_n \sim \theta_n \sim k_n^{-1/3}$. The time evolution of the dissipative system (3.1) and (8.2) is chaotic and confined to a strange attractor in the $4N$-dimensional phase space.

Equations (3.1) and (8.2) are integrated numerically. For the sake of simplicity, we study only the case of the Prandtl number $Pr = \nu/D = 1$. At time $t = 0$, we perturb the system on small scales, by imposing

$$u_n'(0) = u_n(0) + \delta u_n(0) \qquad \theta_n'(0) = \theta_n(0) + \delta\theta_n(0)$$

where $\delta u_n(0) \neq 0$ and $\delta\theta_n(0) \neq 0$ only for $n = n^*$, $n^* + 1$ and n^* corresponds to the Kolmogorov length. The time T^u is defined as the maximum time such that

$$|\delta u_4|^2 + |\delta u_5|^2 < \delta_{\max}$$

and, similarly, T^θ is the maximum time such that

$$|\delta\theta_4|^2 + |\delta\theta_5|^2 < \delta_{\max}.$$

Note that if $\delta u(0) = 0$ then $|\delta\theta(t)|$ does not grow since the equation for θ is linear. On the other hand if $\delta u(0) \neq 0$ one can have exponential growth for $|\delta\theta(t)|$.

In our numerical simulations we considered the case $\delta\theta_n(0) = \delta u_n(0)$. However, the results are not very different in the case $\delta\theta_n(0) = 0$ and $\delta u_n(0) \neq 0$. For $\delta\theta_n(t)$ one has an inverse cascade and a scenario similar to that one of the velocity discussed in section 4.

The PDF of T^θ is similar to that of T^u, i.e. it is sharply peaked at relatively small *Re* and with long exponential tails at large *Re*. We can define the typical exponential growth rate of the temperature perturbation, $\langle 1/T^\theta\rangle$, the 'Lagrangian Lyapunov exponent', and $\langle 1/T^u\rangle$ the 'Eulerian Lyapunov exponent' (Crisanti *et al* 1991). The predictability of the temperature is observed to be shorter than the predictability of the velocity fields, e.g. one has

$$\langle 1/T^\theta\rangle > \langle 1/T^u\rangle.$$

Moreover, T^θ is often much larger than T^u, see figure 5. Large values of T^u, corresponding to the laminar phase for the velocity, can coexist with T^θ close to the typical values.

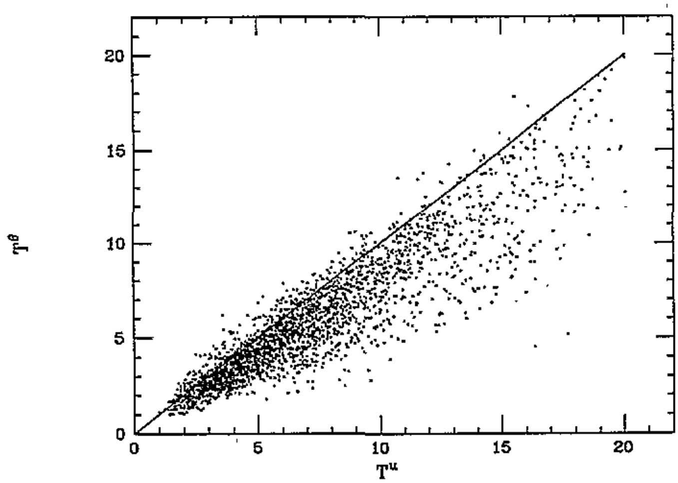

Figure 5. T^θ against T^u for the shell model with 27 shells, $Re = 10^9$, $S = f = 5(1+\mathrm{i}) \times 10^{-3}$ and $K_0 = 0.05$.

This phenomenon is related to Lagrangian chaos (Aref 1984, Crisanti *et al* 1991), and may have some relevance in practical applications, e.g. weather forecasting. In the limit of $D = S = 0$, equation (8.1) is strictly related to the motion of test particles which evolve, driven by the flow, according to the equation

$$\dot{x} = v(x, t). \tag{8.4}$$

Introducing the formal evolution operator of (8.4),

$$x(t) = \mathcal{S}^t\, x(0)$$

$\theta(x, t)$ can be written as

$$\theta(x, t) = \theta_0(\mathcal{S}^{-t}\, x) \tag{8.5}$$

where $\theta_0(x) = \theta(x, t = 0)$. It is well known that (8.4) can exhibit chaotic behaviour, even in the absence of Eulerian turbulence.

Let us consider the evolution of two passive scalars θ and θ' driven by two slightly different velocity fields v and v'. From (8.5) it follows that even if the distance $|v - v'|$ does not grows, i.e. the field is not chaotic, $|\theta - \theta'|$ can grows exponentially in time if (8.4) is chaotic.

In the case of turbulent velocity fields it is not easy to understand the behaviour of a passive scalar in the inertial range. This is due to the non-trivial correlations between energy and passive scalar dissipation. See Jensen *et al* (1992) for a discussion on this point. This makes it difficult to repeat the argument of section 5 for the Lagrangian Lyapunov exponent.

Acknowledgments

We thank M Vergassola for illuminating suggestions and R Benzi and S Ciliberto for useful discussions.

References

Aref H 1984 *J. Fluid Mech.* **143** 1
Anselmet F, Gagne Y, Hopfinger E J and Antonia R 1984 *J. Fluid Mech.* **140** 63
Benettin G, Galgani L, Giorgilli A and Strelcyn J M 1980a *Meccanica* **15** 9
—— 1980b *Meccanica* **15** 20
Benzi R and Carnevale G F 1989 *J. Atmos. Sci.* **46** 3595
Benzi R, Paladin G, Parisi G and Vulpiani A 1984 *J. Phys. A: Math. Gen.* **17** 3521
—— 1985 *J. Phys. A: Math. Gen.* **18** 2157
Burgers J M 1974 *The Nonlinear Diffusion Equation* (Dordrecht: Reidel)
Crisanti A, Falcioni M, Paladin G and Vulpiani A 1991 *Riv. Nuovo Cimento* **14** 1
Crisanti A, Jensen M H, Vulpiani A and Paladin G 1992 *Phys. Rev.* A **46** R7363
—— 1993 *Phys. Rev. Lett.* **70** 166
Desnyansky V N and Novikov E A 1974 *Izv. Akad. Nauk SSSR Fiz. Atmos. Okeana* **10** 127
Eckmann J P and Procaccia I 1986 *Phys. Rev.* A **34** 659
Fujisaka H 1983 *Prog. Theor. Phys.* **70** 1264
Grassberger P, Badii R and Politi A 1988 *J. Stat. Phys.* **51** 135
Jensen M H, Paladin G and Vulpiani A 1991 *Phys. Rev.* A **43** 798
—— 1992 *Phys. Rev.* A **45** 7214
Kida S and Ohkitani K 1992 *Phys. Fluids.* A **4** 1018
Kolmogorov A N 1941 *CR (Dokl.) Acad. Sci. USSR* **30** 301
Laplace S. 1814 *Essai philosophique sur les probabilités* (Paris: Courcier)
Leith C E and Kraichnan R H 1972 *J. Atmos. Sci.* **29** 1041
Lilly D K 1973 *Dynamic Meteorology* ed P Morel (Boston: Reidel)
Lorenz E N 1963 *J. Atmos. Sci.* **20** 130
—— 1969 *Tellus* **21** 3
Mandelbrot B B 1975 *J. Fluid Mech.* **72** 401
Monin A S and Yaglom A M 1975 *Statistical Fluid Mechanics* vol II (Cambridge, MA: MIT)
Oseledec V I 1969 *Trans. Moscow Math. Soc.* **19** 167
Paladin G, Peliti L and Vulpiani A 1986 *J. Phys. A: Math. Gen.* **19** L991
Paladin G and Vulpiani A 1987a *Phys. Rep.* **156** 147
—— 1987b *Phys. Rev.* A **35** 1971
Parisi G and Frisch U 1985 *Turbulence and Predictability of Geophysical Flows and Climatic Dynamics* ed M Ghil, R Benzi and G Parisi (New York: North-Holland)
Pomeau Y and Manneville P 1980 *Commun. Math. Phys.* **74** 149
Ruelle D 1979 *Phys. Lett.* **72A** 81

Vincent A and Meneguzzi M 1991 *J. Fluid Mech.* **225** 1
Yamada M and Ohkitani K 1987 *J. Phys. Soc. Japan* **56** 4210
—— 1988 *Prog. Theor. Phys.* **79** 1265